Different algorithms for calculating the deformations of a point
Zdenko MAZUREK 1*, Milivoj VULIČ 2
1Borisa Kidriča 2, SI-8330 Metlika, Slovenia
2University of Ljubljana, Faculty of Natural Sciences and Engineering, Slovenia

Abstract. Underground excavation makes a very big impact to the ground surface, which is reflected by the deformation on the surface. Deformation of the surface is reflected as the slope, curvature, horizontal or vertical displacement and dilatation, which are very dangerous for the buildings. In this article we will present important forms for calculating the deformation and the analytical methods with which we can calculate different types of the deformation based on the statistical forms. Deformation of the point depends on the direction of the observed profile. The components of the deformation are usually written in strain matrix 2x2 or 3x3 from which we can calculate the main values and their direction. Here we will detaily present different algorithms for calculation the eigenvalues and eigenvectors of the matrix.

Keywords: deformation, analytical method, matrix, eigenvalues, eigenvectors

* zdenko.mazurek@gmail.com
1 Introduction

Deformations on the surface are reflected as a change of the slope, curvature, horizontal or vertical displacements and dilatation. The behaviour of the terrain is dependent by typical influence factors:

- length and width of the underground mine,
- depth of the underground mine,
- tendency of the profile, and
- strength coefficient of the overburden.

We derived the formulas for calculating the slope, curvature and dilatation on the surface. Also, we presented the forms of analytical methods for calculating the deformation on the point, which are based on the statistical models. This method we can easily use with MS Excel™.

2 Definition of the different deformation types

2.1 Slope

Slope is defined with the difference of the subsidence adjacent points depending on their distance. It is expressed in mm/m or % [6]. If the change in the slope occurs, it can be dangerous for higher building.

The simple equation for calculation the slope is showed in the next formula, where we can see that, if the difference of the subsidence increases between two points, the value of the slope also becomes higher [6]:

\[ N = \frac{u_A - u_B}{l_{AB}} \]  

2.2 Curvature

The curves which are not a circle have a different curvature in each point. If the curvature in each point equals \( \frac{1}{R} \), than the curvature in this point is the same as the circle of curvature with radius \( R \).

It is defined as the first curvature or flexion and the second curvature or torsion. The flexion is a plain curvature and the torsion presents its deviation if it is the plain curvature. The plain curve has the torsion curvature equal to zero and the whole curve is located in a plane.

The basic equation of the curvature for \( y = f(x) \) is [10]:

\[ K = \frac{da}{ds} = \frac{y'}{(1+y'^2)^{3/2}} \]  

The curvature is dependent on subsidence of the points in the surface. Subsidence is different in each point and the curvature can be convex or concave. If we know the subsidence in three different points on the surface, the curvature is defined as the difference of the slope depending on the middle length of the points [6]:

\[ K = \frac{u_2-u_1}{l_1} - \frac{u_3-u_2}{l_2} \]

\[ \frac{1}{2(1+l_2)} \]

2.3 Dilatation

Declination

Declination is shown in Figure 3. The points \( A_0 \)

\[ K_{ab} = \frac{\Delta N}{a} \]
and $B_0$, forming the first distance $l_0$ change their position to $A$ and $B$ respectively, which are mapped to the axis $\xi$ and they are labelled as $A'$ and $B'$. Distance between $A'$ and $A$ is labelled as $P_{\eta A}$ and the distance between $B'$ in $B$ is labelled as $P_{\eta B}$ [6].

$$v = \frac{P_{\eta A} - P_{\eta B}}{l_0}$$  

(5)

### Sliding and rotation

In the case of sliding we have to observe the point $A$, $B$ and $C$ on the surface of the non-deformed massive, with the distances $AB = l_1$ and $AC = l_2$ which form the rectangular $\angle BAC = 90^\circ$ [1].

The sliding is present if the points after consolidation change their position to $A_1$, $B_1$, and $C_1$, while, in this case, the angle is not rectangular ($\angle B_1A_1C_1 \neq 90^\circ$) [6].

### Dilatation

Dilatation occurs because of the horizontal displacement of the points on the surface. Dilatation is expressed in % or mm/m. That is a very dangerous type of the deformation, because with the expansion and contraction, it makes huge damages in the buildings.

$$D_\xi = \frac{\Delta l}{l_0}$$  

(8)

### 3 Use of the analytical method based on the statistical forms

Here we give few formulae based on statistical forms for calculation of the slope and the curvature of the surface above an underground mine.

They present the values of the slope and curvature in the selected profile, which could be happen in the mine shaft with the input data shown in the figure. The values are calculated with MS ExcelTM (Table 1).

Equations used for calculation of those values are [6]:

**SLOPE:**

$$N(y) = N_{1(y)} + N_{2(y)}$$  

(9)

with:

$$N_{1(y)} = \frac{qU_0}{\sqrt{[Hcota-y(\varphi(\xi)-\varphi(\eta))]}$$  

(10)

$$N_{2(y)} = \frac{1}{2Hcota-y[\varphi(\xi)-\eta\varphi(\eta)]}$$  

(11)

**CURVATURE:**

$$K(y) = K_1(y) + K_2(y) + K_3(y)$$  

(12)

with:

$$K_1(y) = -\frac{q^2U_0}{[Hcota-y(\varphi(\xi)+\eta\varphi(\eta))]}$$  

(13)

$$K_2(y) = \frac{qU_0}{(Hcota-y)^2[1-(1-\xi^2)\varphi(\xi)+\eta(1-\eta^2)\varphi(\eta)]}$$  

(14)

$$K_3(y) = \frac{qU_0}{(Hcota-y)^2[3-(1-\xi^2)\varphi(\xi)+\eta(3-\eta^2)\varphi(\eta)]}$$  

(15)

The calculated values show interrelation between the slope and curvature. When the slope has extreme values, the values of the curvature are minimal or zero (Figure 4).
In fact, the word “eigen” symmetric matrix 2x2 and 3x3 equations of the plane strain an eigenpair for the matrix $A$

ew{4.1 Definition of the eigenvalue and
determine:}

Deformation of the points is (among other things) “characteristic” in German

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{INPUT DATA:} & \gamma & N \text{[m/m]} & N(y) \text{[mm/m]} & K_1 \text{[m$^3$]} & K_2 \text{[m$^3$]} & K(y) \text{[km$^3$]} \\
\hline
U_{EF} & 2000 \text{ mm} & -800 & 3.07e-05 & 0.03 & 8.66e-07 & -2.20e-07 & 1.22e-08 & 0.001 \\
b & 295 \text{ m} & -700 & 2.94e-04 & 0.29 & 6.63e-06 & -1.24e-06 & 3.44e-08 & 0.005 \\
m & 122 \text{ m} & -600 & 1.75e-03 & 1.75 & 2.76e-05 & -2.82e-06 & -6.33e-09 & 0.025 \\
H & 400 \text{ m} & -500 & 5.60e-03 & 5.60 & 4.39e-05 & 1.83e-06 & 2.38e-07 & 0.045 \\
\alpha & 0.32 \text{ m$^{-1}$} & -417 & 8.07e-03 & 8.07 & 0.00e+00 & 8.17e-06 & 0.00e+00 & 0.008 \\
\theta & 35 \text{ °} & -300 & 3.88e-03 & 3.88 & -5.27e-05 & -2.62e-06 & 2.11e-07 & -0.055 \\
\lambda & -200 & 4.16e-04 & 0.42 & -1.19e-05 & -2.80e-06 & -1.21e-07 & -0.015 \\
\mu & -122 & 0.00e+00 & 0.00 & -7.26e-07 & -2.85e-07 & -2.51e-08 & -0.001 \\
\nu & 0 & -7.51e-04 & -0.75 & -2.30e-05 & -5.05e-06 & -2.29e-07 & -0.029 \\
\xi & 300 & 6.00e-03 & -6.00 & -1.03e-04 & -2.01e-06 & 3.01e-06 & -0.104 \\
\phi & 173 & 1.27e-02 & -12.71 & 0.00e+00 & 3.19e-05 & 0.00e+00 & 0.002 \\
\eta & 250 & -5.57e-03 & -5.57 & 1.35e-04 & -1.50e-05 & 1.18e-06 & 0.119 \\
\theta & 330 & -9.32e-05 & -0.09 & 6.14e-06 & -3.61e-06 & 4.61e-07 & 0.003 \\
\hline
\end{array}
$$

Table 1: The values of the slope and curvature for selected profile

4 The main values of the deformation and
direction

Deformation of the points is often written as the square matrix 2x2 or 3x3. From the strain tensor we can determine:

- Eigenvalues, which present the main values of the deformation,
- Eigenvectors, which present the direction of the each main value.

4.1 Definition of the eigenvalue and the
eigenvector

Let $A$ be a square $n \times n$ matrix. The eigenvector of $A$ is a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that for some scalar $\lambda$, we have:

$$
A\mathbf{v} = \lambda \mathbf{v}. 
$$

(16)

The scalar $\lambda$ is called the eigenvalue of the matrix $A$, and we say that the vector $\mathbf{v}$ is the eigenvector belonging to the eigenvalue $\lambda$. The pair $\{\lambda, \mathbf{v}\}$ is called an eigenpair for the matrix $A$. In fact, the word “eigen” means (among other things) “characteristic” in German [7].

In the continuation of this article we present the different graphics, analytical and numerical methods for calculation of the eigenvalues and the eigenvectors of:

- symmetric matrix 2x2 and 3x3 and
- asymmetric matrix 2x2 and 3x3.

4.2 The scope of the practical part

In the literature, different methods for calculating the eigenvalues and eigenvectors are indicated. Therefore, we present here different algorithms for calculating the eigenvalues and eigenvectors, summarized from various available references.

The beginning of the research was searching for the available references with algorithms to be summarized. After that, the equations were entered into MS Excel™ spreadsheet, which resulted in findings for different methods. We made the criterion for the estimating of each algorithm which resulted in Table 1.

The list of methods used for analyse:

- equations of the plane strain,
- Mohr’s cycle for symmetric and asymmetric matrix,
- the method “Basics of rigid body mechanics” [9],
- the method “Material resistance with the theory of elasticity” [4],
- functions Eigenvalues_pow and Eigenvectors_pow,
- functions matEigenWsymtensorRank2 and matEigenWtensorRank2, and
- Gershgorin circles.

All methods, except Mohr’s cycle and Gershgorin circles, were analysed with MS Excel™.

![Figure 5: Criterion for estimating of the each method](image-url)
5 Methods

5.1 Equations of the plane strain

The equations of this method are not complex and can be easily remembered. It enables very fast calculation of the main deformations and their directions of the symmetric matrix 2x2. This method we could not use for asymmetric matrix.

The relations used for calculating the eigenvalues (main values) [6]:

\[\varepsilon_1 = \frac{1}{2} (\varepsilon_x + \varepsilon_y) + \frac{1}{2} \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + 4\varepsilon_{xy}^2} \quad (17)\]

\[\varepsilon_2 = \frac{1}{2} (\varepsilon_x + \varepsilon_y) - \frac{1}{2} \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + 4\varepsilon_{xy}^2} \quad (18)\]

The equation for calculating the eigenvectors (direction of the main values) [6]:

\[\tan 2\varphi = \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y} \quad (19)\]

5.2 Mohr's circles for symmetric and asymmetric matrix

This method is easy to remember and the construction of the cycle is relatively quick. The Mohr's circles, described in [5,8], get the values of the eigenvalue and eigenvector from symmetric or asymmetric matrix 2x2. With this method we could not get very high precision, especially if we draw the circles by hand.

Steps for construction of the Mohr's circle:

The first step when we constructed the Mohr's circle is to input the coordinates of the deformation components \((\varepsilon_1 xx, \varepsilon_1 xy)\) and \((\varepsilon_1 yy, -\varepsilon_1 xy)\). Than we find the centre C and radius R.

The centre and radius of the Mohr's circle for asymmetric matrix are calculating with the next forms [5]:

- The centre of the circle:
  \[C = \frac{1}{2} \left[ (\varepsilon_{xx} + \varepsilon_{yy}), (\varepsilon_{xy} - \varepsilon_{yx}) \right] \quad (20)\]

- The radius of the circle:
  \[R = \frac{1}{2} \left[ (\varepsilon_{xx} + \varepsilon_{yy})^2 + (\varepsilon_{xy} - \varepsilon_{yx})^2 \right]^{\frac{1}{2}} \quad (21)\]

The main values present the points where the circle is crossing the abscissa.

Figure 6. Mohr's circles for symmetric matrix 2x2 [6]

Figure 7. Mohr's circles for asymmetric matrix 2x2

5.3 The method "Basics of rigid body mechanics"

This method gives us high precision values from symmetric and asymmetric matrix 3x3. This method uses Newton's numerical method for finding the first eigenvalue from the characteristic polynomial of the matrix.

We can find the eigenvalues via Taylor series near the selected proxy [9]:

\[f(\lambda) = F(\lambda_0) + \frac{df}{dx}|_{x_0} (x - x_0) + \frac{df^2}{dx^2}|_{x_0} (x - x_0)^2 + \ldots \quad (22)\]

We keep the first two terms, other we can ignore [9]:

\[f(x) = f(x_0) + f'(x_0)(x - x_0) = 0 \quad (23)\]

With this relation we can calculate the first eigenvalue [9]:

\[x = x_0 - \frac{f(x_0)}{f'(x_0)} \quad (24)\]

given \(x_0\) as the initial guess.

Calculating with this method is passing iterative until reaching the required precision. To get the other eigenvalues, we have to divide the characteristic
polynomial with a monomial \((x - \varepsilon_1)\) [9]. This method is easy to remember but it is not quick especially if we calculate the values manually.

The eigenvectors are calculated on the basis of the system with two equations and two variables [9]. Also, if we want to get a unit vector, we have to normalize the vector:

\[ \mathbf{v}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \quad (25) \]

5.4 The method “Material resistance with the theory of elasticity”

With this method we can calculate the zeroes (main deformations) of the characteristic equations using the cubic equation (the conditions of this method are: \(D < 0\) and \(p < 0\)) [4].

If the characteristic equation is:

\[ a \cdot \varepsilon_i^3 + b \cdot \varepsilon_i^2 + c \cdot \varepsilon_i + d = 0 \quad (26) \]

we introduce the new variable [4]:

\[ y_i = \varepsilon_i + \frac{b}{3a} \quad (27) \]

\[ 2 \cdot q = \frac{2b^3}{27a^3} - \frac{b \cdot c}{3a^2} + \frac{d}{a} \quad (28) \]

\[ r = \sqrt{|p|} \quad (29) \]

In the case of the forms with \(p < 0\) and \(q^2 + p^3 < 0\), we apply the cubic equations [4]:

\[ \cos \varphi = \frac{q}{r^3} \quad (30) \]

\[ y_1 = -2 \cdot r \cdot \frac{\cos \varphi}{3} \quad (31) \]

\[ y_2 = 2 \cdot r \cdot \cos \left(60^\circ + \frac{\varphi}{3}\right) \quad (32) \]

\[ y_3 = 2 \cdot r \cdot \cos \left(60^\circ - \frac{\varphi}{3}\right) \quad (33) \]

Then the eigenvalues are calculated with the relation [4]:

\[ \varepsilon_3 = y_3 - \frac{b}{3} \quad (34) \]

The directions of the main values are calculated by forming the 2x2 determinants \([A_i], [B_i]\) and \([C_i]\). The steps for calculating the first direction \(\alpha_1, \beta_1\) are presented in \(\gamma_1\) [4]:

\[ |A_i| = \begin{pmatrix} (\varepsilon_1 - \varepsilon_1) & \frac{1}{2} \gamma_{yx} \\ \frac{1}{2} \gamma_{xy} & (\varepsilon_1 - \varepsilon_1) \end{pmatrix} \quad (37) \]

\[ |B_i| = \begin{pmatrix} \frac{1}{2} \gamma_{yx} & \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{zx} & (\varepsilon_1 - \varepsilon_1) \end{pmatrix} \quad (38) \]

\[ |C_i| = \begin{pmatrix} \frac{1}{2} \gamma_{xy} & \varepsilon_1 - \varepsilon_1 \\ \frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{xy} \end{pmatrix} \quad (39) \]

with the angle theorem [4]:

\[ \frac{\cos \alpha_1}{|A_i|} = \frac{\cos \beta_1}{|B_i|} = \frac{\cos \gamma_1}{|C_i|} = k \quad (40) \]

\[ \cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1 = 1 \quad (41) \]

\[ k = \frac{1}{\sqrt{|A_i| + |B_i| + |C_i|}} \quad (42) \]

Those steps we have to repeat for the direction \(\alpha_2, \beta_2\) and \(\gamma_2\), as well as for the direction \(\alpha_3, \beta_3\) and \(\gamma_3\).

5.5 Functions Eigenvaule_pow and Eigenvector_pow

The functions are available on the internet [3] as a UDF matrix.xla. Function Eigenvaule_pow gives the eigenvalues of the symmetric or asymmetric matrix 2x2 or 3x3. Function Eigenvector_pow gives the eigenvectors of the symmetric or asymmetric matrix 2x2 or 3x3. These functions use the power method. The power method generates the sequence of vectors \(A^k \mathbf{v}_0\) where \(\mathbf{v}_0\) is a nonzero initial vector. This sequence of vectors, when normalized appropriately, converges to a dominant eigenvector, i.e., an eigenvector associated with the eigenvalue of the largest modulus. With the command IterMax [3] we can generate the maximum number of iterations.
5.6 Functions matEigenWsytensorRank2 and matEigenWtensorRank2

The functions are available on the internet as a UDF eigen tensor rank 2.xla [11]. Function matEigenWsytensorRank2 gives the eigenvalues and eigenvectors of the symmetric matrix 2x2 and 3x3. Function matEigenWtensorRank2 gives the eigenvalues and eigenvectors of the asymmetric matrix 2x2 and 3x3. Also, those functions can show us the precision and number of iterations, which can enable us to see how quickly the values converging to the “real” values.

<table>
<thead>
<tr>
<th>INPUT DATA:</th>
<th>RESULTS:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrika A:</td>
<td>iteration precision</td>
</tr>
<tr>
<td>2 0,5 0,9</td>
<td>103 3.20168E-16</td>
</tr>
<tr>
<td>0,5 1 -1</td>
<td>eigenValue Major eigenValue Semi eigenValue minor</td>
</tr>
<tr>
<td>0,9 -1 0,5</td>
<td>2.422459521 1.747311329 -0.66977085</td>
</tr>
<tr>
<td></td>
<td>eigenVector Major eigenVector Semi eigenVector minor</td>
</tr>
<tr>
<td></td>
<td>0.911521267 -0.20393606 0.357126116</td>
</tr>
<tr>
<td></td>
<td>0.032175463 -0.830359231 -0.55629874</td>
</tr>
<tr>
<td></td>
<td>0.40999234 0.51856883 -0.750328361</td>
</tr>
</tbody>
</table>

5.7 Gershgorin’s circles

It is a very useful method for fast determination of the approximate eigenvalues. From the theory of the Gershgorin’s circles, the main values \(\varepsilon_i\) must be located in the union of the circles. This method does not have a lot of equations and it is easily to remember.

The steps of the construction of the Gershgorin’s circles [1]:

- First we find the centre of the circle:

\[
C = A_{ii}
\]

- We determine the radius of the circle with next equation:

\[
d_i = \sum_{j \neq i} |A_{ij}| \tag{44}
\]

Here we present the Gershgorin’s circles of the asymmetric matrix 3x3. Let us find the Gershgorin’s circles of the matrix \(A\) (Figure 10), which has the eigenvalues: \(\varepsilon_1 = 7.91, \varepsilon_2 = 1.63\) in \(\varepsilon_3 = 3.55\).

\[
A = \begin{bmatrix} 8 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 2 & -3 \end{bmatrix}
\]

Figure 10. Gershgorin’s circles of the asymmetric matrix \(A\)
6 Results

The results of analysis are shown in the Table 2.

Table 2: Rating of the individual method depending on the analysed criterion [8].

<table>
<thead>
<tr>
<th>Used methods</th>
<th>Rating of the individual method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equations of the plane strain</td>
<td>ACCEPTABLE</td>
</tr>
<tr>
<td>Mohr's circles</td>
<td>ACCEPTABLE</td>
</tr>
<tr>
<td>The method “Basics of rigid body mechanics”</td>
<td>WELL</td>
</tr>
<tr>
<td>The method “Material resistance with the theory of elasticity”</td>
<td>WELL</td>
</tr>
<tr>
<td>Functions MatEigenvalue_pow and MatEigenvector_pow</td>
<td>WELL</td>
</tr>
<tr>
<td>Function EigenWsymTensorRank2</td>
<td>WELL</td>
</tr>
<tr>
<td>Function EigenWTensorRank2</td>
<td>WELL</td>
</tr>
<tr>
<td>Gershgorin's circles</td>
<td>ACCEPTABLE</td>
</tr>
</tbody>
</table>

7 Discussion

The intention of this analyse is presenting the usability of analysed methods and easier decision which method we can use for different types of matrix. Results of analysed methods are self-explainable. The reason of that is because every method is evaluated with the criterion which is described in the scope of the particular part. If we use the different criterion or if we change the existing criterion, the results of the analysed methods will be different. It is possible to perform further analysis of these (and a lot of other) methods, and then we can update and adjust the existing criterion.

**Equation of the plane strain** is the best method when we must calculate the symmetric matrix 2x2. This method is rated “acceptable”, because this method can calculate only symmetric matrix 2x2.

**Mohr's circles** method is a very easy and quick graphical method. The biggest problem of the Mohr's circles is its less precision in comparison with the other calculating methods. Because of that, this method is rated “acceptable”.

The method “Basics of rigid body mechanics” calculates the symmetric and asymmetric matrix 3x3. The method is relative easy to remember, but calculating with this method is not very quick. Also, there is a presence of an increased risk for making mistake, especially when we derive the characteristic equations with their monomial. In this method we can choose the initial guess, what could be its advantage. This method is rated “well”.

The method “Material resistance with the theory of elasticity” calculates the symmetric matrix 2x2 and 3x3. This method is hard to remember because there are more different analytical forms in comparison with method “Basics of rigid body mechanics”. That method is rated “well”.

Functions **MatEigenvalue_pow** and **MatEigenvector_pow** calculate the symmetric and asymmetric matrix 2x2 and 3x3. We could not get the initial guess, but only the maximum number of iterations. The functions are rated “well”, because they also do not show the precision and the number of iterations.

Functions **eigenWsymTensorRank2** and **eigenWTensorRank2** calculate the symmetric and asymmetric matrix 2x2 and 3x3. We can fast remember the steps of these functions– mnemonically. The functions give the precision and the number of iterations so we can see how quickly the values are converging to real values which is the advantage of this method, because it gives us, at the same time, both the eigenvalues and the eigenvectors, but this can be also the disadvantage if we need only one of those values. We could not give this method the initial guess which would faster converge to the real value. This method is rated “well”.

**Gershgorin circles** we can use for symmetric and asymmetric matrix 2x2 and 3x3. This method can be very interesting in the case if we need the initial value for beginning the calculation of the eigenvalues with one of the iterative method. This method is rated “acceptable”.

8 Conclusion

The decision which method we will use to calculate the eigenvalues and the eigenvectors depends on the dimension and symmetry of the matrix. Some of the methods are more complex than other, so before starting with the calculation, we have to decide which method is more practical to use (example: if we have symmetric matrix 3x3, than is more practical if we use the function **eigenWsymTensorRank2** than **eigenWTensorRank2**). The analytical methods are less successful than numerical and iterative methods, which are strong mathematical tools for calculation of the requested values.

References

