Comparison of height differences obtained by trigonometric and spirit leveling method

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Abstract. This paper aims to show some research about comparison of heights differences obtained by trigonometric and spirit leveling method. Even though it seems that height differences obtained by these two methods could not be compared it is possible to state that, under some conditions, it is possible to compare them. Bearing in mind the essential characteristics of trigonometric and spirit leveling it is possible, under the some restrictions, to state that heights differences obtained by trigonometric and spirit leveling are comparable. Those limitations are firstly related to the trigonometric leveling method because the different influences considerably limit its accuracy. In that sense the results of height differences obtained by spirit leveling method could be considered as significantly more accurate than the results obtained by trigonometric leveling method. In this paper some short analysis of both methods for heights differences determination are given and, after that, the case study from the practice is shown and analyzed according to the given and utilized mathematical and statistical model.

Key words: statistical hypothesis testing, pairs of measurement, trigonometric leveling, spirit leveling.

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Introduction

The main difference between trigonometric and spirit levelling method for height difference determination is in the construction of geodetic instruments used for them, in the measurement methods and in the influences which affects the accuracy. Those differences significantly limit the area in which comparison of these two methods is possible. This research is limited on the line levelling length of few hundred meters and also lines of sight shorter than 300 m.

Spirit levelling is based on the fact that axis of spirit level is perpendicular on the plumb line and height difference between two points is obtained as difference of readings obtained on the level rods settled on those two points. The level position is considered in the middle of those two points. Detailed analysis of heights differences obtained by spirit levelling is given in [3], while detailed analysis of spirit levelling method is given in [4], [5]. According to given literature immediately follows that spirit levelling method is very accurate method especially for short levelling lines. However, necessity of gravimetric measurements for correct interpretation of height differences obtained by spirit levelling, significantly limit the possibilities for their comparison when they are not measured at the same paths.

Trigonometric levelling method is based on measurements of distances, zenith angles and used trigonometric function for height differences determination between two points. Height difference as a complex function of different measurements (distance and zenith angle) and often unforeseen parameters introduces errors which are not common for spirit levelling. Especially influence on the accuracy of height differences obtained by trigonometric levelling and possible major limitation for its accuracy is refraction which limits the accuracy of zenith angle. Also the error of measured distance is considerable limit for accuracy of height difference obtained by trigonometric levelling method even in the case when most accurate distance meters were used [7]. And finally, the form of mathematical model (trigonometric function) used for height difference determination cause certain error according to law of error propagation [7].

In this paper one effort was done aiming to show possibilities of comparison the same height differences obtained by spirit levelling and by trigonometric levelling. Firstly suitable mathematical and stochastic models will be researched and after that the theoretical approach will be utilized on the measured values.

2 Background

Height differences by spirit levelling are obtained as the differences of read values on the two levelling rods:

\[ \delta H_B^A = l_B - l_A \]

where \( l_B \) and \( l_A \) are values read on the levelling rods put on benchmarks B and A. Because equipotential surfaces are non-parallel only geometric interpretation of height differences obtained by spirit levelling method is not absolutely correct and for their proper interpretation it is necessary to have gravimetric measurements as shown on figure 1 [3].

![Figure 1. Heights obtained by spirit levelling and orthometric height](image-url)

There are different definitions of trigonometric levelling [7] but one of them include triangulation principles and trigonometric calculations needed for height difference determination [8]. This definition could be widely used because of various models for height differences determination by using trigonometric levelling [6], [7] and [9]. Common formula for height differences obtained by
trigonometric levelling method could be expressed as:
\[ h = \varphi(H, s, Z, k, r) + i - l \] \hspace{1cm} (1)
\[ h = \vartheta(H, D, Z, k, r) + i - l \] \hspace{1cm} (2)

where:
\( h \) - height difference obtained by trigonometric levelling;
\( H \) - height of station point above the sea;
\( s \) - slope distance from the centre of telescope to the measured point;
\( Z \) - zenith angle;
\( D \) - horizontal distance from the centre of telescope to the measured point;
\( k \) - air refraction coefficient;
\( r \) - mean radius of the Earth;
\( i \) - height of telescope centre above the point and \( l \) - height of measured point.

The terms \( i \) and \( l \) could be eliminated from the equations (1) and (2) if accepted \( i = 0 \) and \( l = 0 \), i.e. if telescope centre and measured point are the points between which the height difference shall be determined.

Functions \( \varphi(s, Z, k) \) and \( \vartheta(D, Z, k) \) get the following forms respectively [6], [10]:
\[ \varphi(s, Z, k) = scosZ + \frac{1-k}{2r}S^2\sin^2Z \] \hspace{1cm} (3)
\[ \vartheta(D, Z, k) = DctgZ + \frac{1-k}{2r} D^2 + DctgZ \frac{H_m}{r} \] \hspace{1cm} (4)

where \( H_m \) is the average height of station point and measured point.

In [9] following formulae are used for trigonometric height differences determination:
\[ h = X + B * Y^2 \] \hspace{1cm} (5)
where:
\[ X = scos\zeta \]
\[ Y = s|\sin\zeta| \]
\[ B = \frac{1-k}{2R} = 6.83 * 10^{-8}[\text{m}^{-1}] \]
\[ k = 0.13 \]
\[ R = 6.37 * 10^6 \text{ m} \]

The final form of formula (5) reads:
\[ h = scos\zeta + 6.83 * 10^{-8} * [s|\sin\zeta|]^2 \] \hspace{1cm} (6)
or using the same insignia as given in formulae (1) and (2):
\[ h = scosZ + 6.83 * 10^{-8} * [s|\sinZ|^2] \] \hspace{1cm} (7)

Graphical interpretation of trigonometric levelling is shown on figure 2.
In formulae (3) and (4) the vertical deflection is neglected which affects the height differences accuracy. In [3] the model for trigonometric levelling which includes influence of vertical deflection is given, without influence of air refraction which is considered as known:

\[
h_2 - h_1 = s \left(1 + \frac{h_m}{R} + \frac{s^2}{12R^2}\right) \tan \frac{z_2 - z_1}{2}
\]

where:

\[
z_1 = z'_1 + \varepsilon_1 = z'_1 + \xi_1 \cos \alpha + \eta_1 \sin \alpha \tag{9}
\]

\[
z_2 = z'_2 - \varepsilon_2 = z'_2 - \xi_2 \cos \alpha - \eta_2 \sin \alpha \tag{10}
\]

where \(\xi_1, \eta_1, \xi_2, \eta_2\) are components of vertical deflection on the points 1 and 2 respectively. In this model the air refraction is assumed as known.

For one side height difference determination by trigonometric levelling method and taking the influence of vertical deflection i.e. corrected value of zenith angle \(Z\) instead of measured value of zenith angle \(Z'\). In that case formula (7) becomes:

\[
h = scos(Z + \varepsilon) + B \ast [s \sin (Z + \varepsilon)]^2 \tag{11}
\]

Even though the significant differences between two described methods for heights differences determination does exist especially in measurement method, influences which affect the accuracy and obtained values it is possible to compare the results obtained by those two methods [7]. The main principles which guided this possibility were based on following presumptions:

- If levelling line and distance from station point and measurement points were short the influence of vertical deflections on height differences obtained by spirit and trigonometric levelling shall be insignificant;
- If time interval between measurement of distances and zenith angles between two close points was short the changes of air refraction angle shall be insignificant and
- If the differences of distance and zenith angles between two close points were small the changes of air refraction angle shall be insignificant.

Short analysis of vertical deflections and air refraction influences on trigonometric height differences could be performed by varying the values of them. For accepted values:

- \(\xi = 10''\);
- \(\eta = 10''\);
- \(k = (0.09; 0.13; 0.17)\)
- \(s \in (50 \text{ m,} 300 \text{ m)}\);
- \(\alpha \in (\alpha_0, \alpha_0 + \Delta \alpha)\) and
- \(Z \in (Z_0, Z_0 + \Delta Z)\)

Changes of height differences influenced by vertical deflection and variation of air refraction coefficient are discussed as follows. Even though the components of vertical deflections could be relatively high, only relative difference of plumb lines between level and total station will affect the height differences obtained by two methods. Thus, if we consider relatively small area, i.e. relatively short distance between level and total station, we can write formula (9) in form:

\[
e_i = z'_i + \Delta \xi_i \cos \alpha_i + \Delta \eta_i \sin \alpha_i \tag{12}
\]

where \(\Delta\) means the increment of vertical deflection components for total station and level.

If accepted \(\Delta \xi = \Delta \eta = 1''\) and \(\Delta \alpha = 11.25^\circ\), \(\alpha \in (0, 360^\circ)\) the changes of vertical deflections \(\varepsilon\) with changes of azimuth are shown on diagram 1.

![Diagram 1. Changes of vertical deflection with azimuth](image-url)
As shown on diagram 1, increment of vertical deflection between two close points (in mean of azimuth) is small (less than 0.28") for azimuth increment \(\Delta \sigma = 11.25^\circ\). Vertical deflection increment can only be decreased if increment of azimuth decreases.

Influence of air refraction coefficient is researched on the interval \(k \in (0.09; 0.17)\) because it is assumed that small changes in azimuth, time interval of measurement, changes of zenith angle and distance between close points shall not change it more than accepted value. The changes of height differences influenced by changes of air refraction coefficients are shown in table 1. As read in table 1 it is possible to state that big changes in air coefficient refraction influence the height differences almost insignificantly if slope distances are shorter than 100 m and it is about 0.0001 m, and if difference of distance between two close points is less than 50 m.

<table>
<thead>
<tr>
<th>D [m]</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta h) [m]</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0004</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

Bearing in mind that height difference between two points \(i\) and \(j\) is obtained by following formulae:

\[
h_i = s_i \cos(Z_i + \varepsilon_i) + B_i \cdot [s_i \sin(Z_i + \varepsilon_i)]^2
\]

\[
h_j = s_j \cos(Z_j + \varepsilon_j) + B_j \cdot [s_j \sin(Z_j + \varepsilon_j)]^2
\]

\[
\Delta h_i^j = h_j - h_i
\]  
(13)

it could be expected that some of influences will be decreased.

Statistical hypothesis testing about equality of height differences obtained by two different methods are based on the method of pairs of measurements analysis [1]. The statistical hypotheses are formulated for equality of height differences as well as the existence of gross and systematic errors in differences between height differences obtained by spirit and trigonometric levelling method.

3 Results and Discussion

Case study is based on measurement provided on dam “Vrla 2” which is a part of hydropower system “Vlasinske HE” [11]. Dam “Vrla 2” is an earth filled dam located on the river Vrla (south-east Serbia near Sur dulica) which is monitored by geodetic methods twice per year since 2006. For this research 48 benchmarks are used. Between benchmarks the spirit levelling method was provided for height differences determination. Simultaneously, for horizontal coordinate determination the measurement of horizontal directions, slope distances and zenith angles was provided. Because distance and zenith angles were measured by total station of high performance measuring characteristics \((\sigma_z = 0.5^\circ; \sigma_s = 0.5 \text{ mm for } s \leq 120 \text{ m})\) it was possible to compare the difference between height differences obtained by spirit and trigonometric levelling method.

On figure 3 the drawing of levelling network with measured distances and zenith angles from point 111 to the points (benchmarks) is given.

This research did not include vertical deflection and air refraction coefficient determination. Any deviations from presumed values of vertical deflection and air refraction coefficient will decrease accuracy of height differences obtained by trigonometric levelling method.
3.1. Description of measurement and results

Levelling network is consisted of 48 benchmarks and located as shown on figure 3. Measurements for height differences determination were performed by spirit levelling method using the levels and rods with declared accuracy of $\sigma_h = 0.3 \frac{\text{mm}}{\sqrt{\text{km}}}$. Measurements in levelling network were finished in one working day.

Measurements of slope distances and zenith angles were performed from the point 111 (fig.3) on the prism which is put into the hole of every benchmark of levelling network. During the measurement the air temperature, air pressure and air moisture is recorded and used for distance calculation according to formulae given in [9]. For height difference calculation formula (7) is used. In table 2 the height differences obtained by spirit (column $h_{SL}$) and trigonometric levelling (column $h_{TL}$) are given as well as the differences between them (column $d=h_{TL}-h_{SL}$).

Table 2: Height differences between benchmarks obtained by trigonometric and spirit levelling and differences between them

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>$h_{TL}$ [m]</th>
<th>$h_{SL}$ [m]</th>
<th>$d=h_{TL}-h_{SL}$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.1214</td>
<td>0.1218</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.0483</td>
<td>0.0489</td>
<td>0.6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.1069</td>
<td>0.1064</td>
<td>-0.5</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>0.0729</td>
<td>0.0721</td>
<td>-0.8</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>-0.0989</td>
<td>-0.0985</td>
<td>0.4</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>-0.0928</td>
<td>-0.0932</td>
<td>-0.4</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>-0.0910</td>
<td>-0.0924</td>
<td>-1.4</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>-10.3036</td>
<td>-10.3103</td>
<td>-6.7</td>
</tr>
<tr>
<td>8</td>
<td>11</td>
<td>-10.4036</td>
<td>-10.4056</td>
<td>-2.0</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>-0.0877</td>
<td>-0.0879</td>
<td>-0.2</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>0.1172</td>
<td>0.1165</td>
<td>-0.7</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>-0.1433</td>
<td>-0.1424</td>
<td>0.9</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>0.0141</td>
<td>0.0134</td>
<td>-0.6</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>0.1329</td>
<td>0.1306</td>
<td>-2.3</td>
</tr>
<tr>
<td>16</td>
<td>19</td>
<td>-5.0407</td>
<td>-5.0381</td>
<td>2.7</td>
</tr>
<tr>
<td>11</td>
<td>21</td>
<td>-5.4664</td>
<td>-5.4682</td>
<td>-1.8</td>
</tr>
<tr>
<td>19</td>
<td>20</td>
<td>-0.2740</td>
<td>-0.2738</td>
<td>0.2</td>
</tr>
<tr>
<td>20</td>
<td>21</td>
<td>-0.1849</td>
<td>-0.1867</td>
<td>-1.8</td>
</tr>
<tr>
<td>8</td>
<td>122</td>
<td>0.9514</td>
<td>0.9517</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Before further research of data the differences $d$ were tested by Shapiro-Wilk test [1] and null hypothesis is adopted for significance level $\alpha = 0.05$ because $W=0.956>0.941=W_{0.95;41}$.

### 3.2 Obtained Results and Discussion

The following hypotheses were tested through the iterations of analysis:

- Ho: the results of height differences obtained by spirit and trigonometric levelling method are equal;
- Ho: differences between height differences obtained by spirit and trigonometric levelling contain gross errors;
- Ho: differences between height differences obtained by spirit and trigonometric levelling do not contain systematic errors and
- Ha: otherwise in all cases.

For all hypotheses the significance level $\alpha = 0.05$ was adopted. After results calculations according to the method of pairs of measurements [1] the results as following are obtained:

- After 11 iterations and elimination of the same number of gross errors (11 times the Ho about existing gross errors was adopted) there were 30 results for which the hypothesis about equality of height differences obtained by spirit and trigonometric levelling is adopted;
- Also after elimination results which contain gross errors the hypothesis about absence of systematic errors is adopted and
- The mean square error obtained from analysis is $m_h = 0.98$ mm.

Bearing in mind that sample for this research was relatively small the elimination of 11 results is considerable (23% of results were considered as containing gross errors). This fact implies that some influences were present during the measurements. Counter to that, accepting the fact that the sample was extracted from production measurement for deformation analysis of dam without extra efforts made to increase the accuracy of trigonometric levelling results, it is possible to state that some additional analysis and proposals for measurement of distance and zenith angles could improve the accuracy of trigonometric levelling method.

Also the formula (7) (i.e. model given in [9]) uses coefficient of air refraction as a given value in spite of fact that it could vary in larger interval [2].

### 4 Conclusions

Comparison of height differences obtained by spirit and trigonometric levelling methods implies that there exist some influences which limit the accuracy of trigonometric levelling method in spite of the fact that the most accurate instruments were used in presented case study. This result, however (bearing in
mind that present analysis was based on production measurement), indicates that there is possibility for further research for improving accuracy of trigonometric levelling. Especially because the obtained value of mean square error was less than 1 mm for one sided determination of height differences.

Further research shall be devoted to the determination of vertical deflection and air refraction influence on trigonometric height determination by trigonometric levelling method.

References